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Analysis of clipped photocount autocorrelation formulae for nongaussian light

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Abstract. Three models of nongaussian stationary light are studied in which the clipped photocount autocorrelation formulae at zero photon number are not simply connected to the unclipped autocorrelation intensity function of the light field.

1. Introduction

In recent years several methods of measuring correlation functions of fluctuating light fields have been developed: correlation (Hanbury Brown and Twiss 1957), coincidence (Morgan and Mandel 1966), time-to-amplitude converter (Scarl 1968) techniques and photon-counting methods (Arecchi *et al* 1966, Meltzer and Mandel 1970). More recently Jakeman and Pike (1969) have developed theoretically, and then experimentally (Foord *et al* 1970), a new digital correlation method based on properties of clipped photon-counting fluctuations of gaussian light (VanVleck and Middleton 1966). The procedure of clipping has, of course, a great experimental advantage due to the fact that clipping allows the use of a simple electronic circuit, but the theoretical relations connecting the clipped and unclipped autocorrelation functions are generally complicated except in the particular case of a gaussian optical field for which many properties and results are known (Jakeman and Pike 1969). However, it is well known that in some experimental situations (Jakeman *et al* 1970, Picinbono and Rousseau 1970, Schaeffer and Pusey 1972) optical fields of nongaussian nature are present and it becomes very interesting to study the exact theoretical expressions of the autocorrelation functions which are measured in the clipped photocount experiments.

The main object of this paper is, therefore, to extend the clipped photocount autocorrelation formulae at zero photon number to three very simple models of nongaussian optical fields. First, the fundamental formulae of the clipping technique are briefly summarized and then these basic results are applied to sinusoidal and gaussian modulation of the ideal laser light and also to gaussian modulation of thermal light.

2. Fundamental formulae

We briefly recall some fundamental results concerning the clipping technique which have been extensively studied by Jakeman and Pike. For single clipping the appropriate

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correlation function $g_0(\tau)$ takes the form

$$g_0(\tau) = \frac{\langle n_0(0)n(\tau)\rangle}{\langle n_0\rangle\langle n\rangle} = \frac{1}{\langle n_0\rangle\langle n\rangle}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} np(n, m) \tag{1}$$

where

$$n_0(t) = 1 \qquad \text{if} \quad n(t) > 0$$

$$n_0(t) = 0 \qquad \text{if} \quad n(t) = 0.$$

Here $n(t)$ is the number of photoelectrons detected during the time interval T centred about t (very small compared with the coherence time τ_c of the field) and $\langle n\rangle$ is the mean number of photoelectrons. In equation (1), $p(n, m)$ means the joint photon-counting distribution and may be defined as follows:

$$p(n, m) = \frac{(-1)^{n+m}}{n!m!}\frac{\mathrm{d}^n}{\mathrm{d}s_1^n}\frac{\mathrm{d}^m}{\mathrm{d}s_2^m}Q(s_1, s_2)\bigg|_{s_1=s_2=1} \tag{2}$$

with $Q(s_1, s_2)$, the two-dimensional generating function of the intensity $I(t)$, given by

$$Q(s_1, s_2) = \langle\exp(-s_1 I(0)T - s_2 I(\tau)T)\rangle. \tag{3}$$

Here τ is the time delay (arbitrary compared with the coherence time τ_c).

The one-dimensional density probability $p(n)$ that n photoelectrons are registered in a time interval T is given by Mandel's formula (Mandel 1958)

$$p(n) = (n!)^{-1}\int_0^{\infty}(\alpha IT)^n\,\mathrm{e}^{-\alpha IT}p(I)\,\mathrm{d}I \tag{4}$$

where $p(I)$ is the probability distribution of the light intensity $I(t)$ and α is the quantum efficiency of the detector. One can show (Jakeman and Pike 1969) that equation (1) may be written as:

$$g_0(\tau) = \frac{1}{\langle n_0\rangle\langle n\rangle}\left(\langle n\rangle + \frac{\mathrm{d}}{\mathrm{d}s_1}Q(s_1 - 1, s_2)\bigg|_{s_1=s_2=1}\right). \tag{5}$$

Then, after a little calculation, one obtains

$$g_0(\tau) = \frac{1 - q(0, \tau)}{1 - p(0)} \tag{6}$$

where

$$q(n, \tau) = \frac{1}{\langle I\rangle}\left\langle I(0)(I(\tau))^n T^n \frac{\mathrm{e}^{-I(\tau)T}}{n!}\right\rangle. \tag{7}$$

$q(n, \tau)$ is called the 'delayed-triggered' photocounting distribution (Picinbono and Rousseau 1970) and $\langle I\rangle$ means the average value of the intensity $I(t)$. Consider now the formula similar to (5) in the two-channel clipping case. The relevant expression is

$$g_{00}(\tau) = \frac{\langle n_0(0)n_0(\tau)\rangle}{\langle n_0\rangle^2}. \tag{8}$$

So that, we find

$$g_{00}(\tau) = \frac{1}{\langle n_0\rangle^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} p(n, m) \tag{9}$$

which, after a little manipulation, takes the following form:

$$g_{00}(\tau) = \frac{1-2p(0)+p(0,0)}{(1-p(0))^2}. \tag{10}$$

3. Applications

We consider here three kinds of light that can be obtained by modulating cw gas laser light or pseudothermal light. The modulator might consist of an electro-optic KDP crystal mounted between crossed polarizers and with an appropriate applied sinusoidal or noise voltage. It is well known that the fraction of incident light intensity I_0 which is transmitted by an electro-optic light modulator at an arbitrary voltage $V(t)$ can be written as (Yariv 1968)

$$I(t) = I_0 \sin^2 \frac{\pi}{2} \frac{V(t)}{V_0} \tag{11}$$

or

$$I(t) = I_0 \sin^2 bV(t) \tag{12}$$

where V_0 is the characteristic voltage of the crystal modulator and $b = \pi/2V_0$. Moreover if $V \ll V_0$, we obtain

$$I(t) \simeq aI_0 V^2(t) \tag{13}$$

where $a = \pi^2/4V_0^2$ is a constant.

3.1. Sinusoidal modulation of laser light

We describe the instantaneous intensity $I(t)$ by the expression

$$I(t) = 2\langle I\rangle \sin^2(\omega t+\phi). \tag{14}$$

ϕ is the random phase with

$$p(\phi)\,\mathrm{d}\phi = \frac{\mathrm{d}\phi}{2\pi}$$

and

$$\langle I\rangle = \frac{\pi^2}{8}\frac{I_0}{V_0^2}.$$

Using Mandel's formula (equation (4)) with

$$p(I) = \frac{1}{2\pi}\frac{1}{\sqrt{I}}\frac{1}{\sqrt{2\langle I\rangle - I}}, \tag{15}$$

it can be shown that

$$p(n) = \frac{\langle n\rangle^n}{n!} \mathrm{e}^{-\langle n\rangle} \sum_{p=0}^{n} (-1)^p \binom{n}{p} 2^{-P} \sum_{k=0}^{P} \binom{P}{K} I_{|2K-p|}(\langle n\rangle). \tag{16}$$

Using the definition (equation (7)) of $q(n, \tau)$, we then obtain

$$q(n, \tau) = 2(\sin^2 \omega t)p(n) + \frac{n+1}{\langle n \rangle}(\cos 2\omega t)p(n+1), \tag{17}$$

so that, for $g_0(\tau)$ given by equation (6)

$$g_0(\tau) = (1 - e^{-\langle n \rangle} I_0(\langle n \rangle))^{-1}[1 - e^{-\langle n \rangle}\{I_0(\langle n \rangle) - (\cos 2\,\omega\tau) I_1(\langle n \rangle)\}]. \tag{18}$$

$I_0(x)$ and $I_1(x)$ are the zero-order and the first-order Bessel functions of the first kind and $\langle n \rangle$ is the mean value of n.

Likewise, considering the double-clipping formula and taking the $I(t)$ described by equations (14)–(15), we easily find

$$g_{00}(\tau) = 1 + e^{-2\langle n \rangle} \frac{I_0(2\langle n \rangle \cos \omega\tau) - I_0^2(\langle n \rangle)}{(1 - e^{-\langle n \rangle} I_0(\langle n \rangle))^2}. \tag{19}$$

Now, let $\Gamma_I(\tau)$ be the normalized unclipped autocorrelation function of $I(t)$. The straightforward application of equation (11) gives

$$\Gamma_I(\tau) = \frac{\langle I(t)I(t-\tau)\rangle}{\langle I \rangle^2} = 1 + \tfrac{1}{2}\cos 2\omega\tau. \tag{20}$$

With a brief examination of equations (18) and (19), we can conclude that these expressions are not very closely related to (20).

However, it can be seen that the expected limits for $\langle n \rangle \to 0$ of (18) and (19) are

$$\lim_{\langle n \rangle \to 0} \begin{pmatrix} g_0(\tau) \\ g_{00}(\tau) \end{pmatrix} = \Gamma_I(\tau).$$

3.2. Gaussian modulation of laser light

The basic results for a 'real gaussian' field from the real, stationary and gaussian amplitude model of Picinbono (Picinbono and Rousseau 1970) may be summarized as follows:

$$I(t) = I_0 V^2(t) \tag{21}$$

where $V(t)$ is the modulation voltage with gaussian distribution

$$p(V) = \frac{1}{\sigma_V \sqrt{2\pi}} \exp\left(-\frac{V^2}{2\sigma_V^2}\right), \tag{22}$$

σ_V is the variance of $V(t)$ and I_0 is a constant. Thus, with

$$p(I) = \frac{1}{\sqrt{2\pi I \langle I \rangle}} \exp\left(-\frac{I}{2\langle I \rangle}\right) \tag{23}$$

and with the aid of equation (4), one obtains

$$p(n) = \frac{(2n-1)!!}{n!} \frac{\langle n \rangle^n}{(1 + 2\langle n \rangle)^{n+\frac{1}{2}}}. \tag{24}$$

We can extend these for our appropriate calculations and we easily find

$$q(n, \tau) = p(n)\left(1 + 2\frac{n - \langle n \rangle}{1 + 2\langle n \rangle} r_V^2(\tau)\right) \tag{25}$$

and

$$g_0(\tau) = 1 + \frac{2\langle n\rangle}{(1+2\langle n\rangle)^{3/2}\{1-(1+2\langle n\rangle)^{-1/2}\}} r_V^2(\tau) \tag{26}$$

where $r_V(\tau)$ is the normalized autocorrelation function of $V(t)$. For the double clipping formula, we obtain

$$g_{00}(\tau) = 1 + \tfrac{1}{2}\{1+\langle n\rangle-(1+2\langle n\rangle)^{1/2}\}^{-1}\left[\left\{1-\left(\frac{2\langle n\rangle r_V(\tau)}{1+2\langle n\rangle}\right)^2\right\}^{-1/2}-1\right]. \tag{27}$$

On the other hand, the unclipped normalized autocorrelation function of the intensity $I(t)$ is easily derived from equation (21). So we have

$$\Gamma_I(\tau) = 1 + 2r_V^2(\tau). \tag{28}$$

The fundamental relation is, obviously, preserved in the limit

$$\lim_{\langle n\rangle \to 0} \begin{pmatrix} g_0(\tau) \\ g_{00}(\tau) \end{pmatrix} = 1 + 2r_V^2(\tau).$$

3.3. *Gaussian modulation of thermal light*

Let us now consider the intensity

$$I(t) = j(t)V^2(t) \tag{29}$$

where $j(t)$ is a thermal intensity and $V(t)$ is a gaussian modulation with a distribution given by equation (22).

From (29), we can easily deduce the normalized unclipped autocorrelation function of $I(t)$

$$\Gamma_I(\tau) = (1+|\gamma_z(\tau)|^2)(1+2r_V^2(\tau)) \tag{30}$$

where $\gamma_z(\tau)$ is the normalized autocorrelation function of the thermal field and $r_V(\tau)$ is the normalized autocorrelation function of $V(t)$.

Using the expressions for $p(I)$ and $p(n)$ given by (Bendjaballah and Perrot 1971), we firstly obtain for the single-clipping case

$$\begin{aligned} g_0(\tau) = \frac{1-r_V^2(\tau)}{\sqrt{2}} \exp\left(\frac{1}{4\langle n\rangle}\right)\Bigg\{ & \left(\frac{1}{2\langle n\rangle}\right)^{1/4} W_{-1/4,-1/4} \\ & + \frac{r_V^2(\tau)}{2(1-r_V^2(\tau))}\left(\frac{2}{\langle n\rangle}\right)^{3/4} W_{-3/4,1/4} - \frac{|\gamma_z(\tau)|^2}{2}\left(\frac{2}{\langle n\rangle}\right)^{1/4} W_{-5/4,-1/4} \\ & - \frac{3}{4}\left(\frac{2}{\langle n\rangle}\right)^{3/4} \frac{|\gamma_z(\tau)|^2}{1-r_V^2(\tau)} r_V^2(\tau) W_{-1/4,1/4}\Bigg\} \end{aligned} \tag{31}$$

where $W_{\lambda,\mu}$ means $W_{\lambda,\mu}(1/2\langle n\rangle)$ which is the Whittaker function.

To study equation (10) in the present case, we need the expression for $p(n, m)$ which presents some difficulties. To simplify the calculations, we first examine the case where $r_V(\tau) \simeq 1$ which represents the most usual experimental situation (the fluctuations of $V(t)$ are much slower than the optical field fluctuations). Then we have

$$p(0) = (\langle n\rangle)^{-1/2} \exp\left(\frac{1}{4\langle n\rangle}\right) D_{-1}((\langle n\rangle)^{-1/2}) \tag{32}$$

where $D_\lambda(x)$ is the parabolic cylinder function (see for example Gradshteyn and Ryzhik 1965, p 1064). The expression for $p(0,0)$ needed before we can use equation (10) is established in Bendjaballah (1972). It is

$$p(0,0) = \frac{\langle n\rangle^{-1/4}}{2^{5/4}|\gamma_z(\tau)|}\left\{\left(\frac{1}{1+|\gamma_z|}\right)^{-3/4}\exp\left(\frac{1}{4\langle n\rangle(1+|\gamma_z|)}\right)W_{-1/4,-1/4}\left(\frac{1}{2\langle n\rangle(1+|\gamma_z|)}\right)\right.$$
$$\left.-\left(\frac{1}{1-|\gamma_z|}\right)^{-3/4}\exp\left(\frac{1}{4\langle n\rangle(1-|\gamma_z|)}\right)W_{-1/4,-1/4}\left(\frac{1}{\langle n\rangle(1-|\gamma_z|)}\right)\right\}. \tag{33}$$

Thus we can evaluate the full expression for $g_{00}(\tau)$ in (10).

Therefore, if we consider the case $\langle n\rangle \simeq 0$ and use the asymptotic form for $D_\lambda(x)$ and $W_{\lambda,\mu}(x)$ ($x \to \infty$) (see Gradshteyn and Ryzhik 1965, p 1061)

$$p(0) \sim 1-\langle n\rangle+3\langle n\rangle^2$$
$$p(0,0) \sim 1-2\langle n\rangle+3\langle n\rangle^2(3+|\gamma_z(\tau)|^2),$$

we obtain

$$\lim_{\langle n\rangle\to 0} g_0(\tau) = \Gamma_I(\tau)$$

and

$$\lim_{\langle n\rangle\to 0} g_{00}(\tau) = 3(1+|\gamma_z(\tau)|^2).$$

If we use the assumption $\gamma_z(\tau) = 1$, rather than $r_V(\tau) = 1$, we obtain

$$\lim_{\langle n\rangle\to 0} g_{00}(\tau) = 2(1+2r_V^2(\tau)).$$

4. Discussion

In the three models of the light field considered above, the calculations show that the relations which determine the autocorrelation function of the light intensity from single- and double-clipped formulae are generally not simply related. We reach that conclusion by comparing equations (18), (19) with (20), equations (26), (27) with (28) and equations (31), (32), (33), associated with (10), with expression (30).

Of course, the object of clipped experiments is not to give the $\Gamma_I(\tau)$ autocorrelation function of the light intensity directly but to lead to the exact autocorrelation function of the light intensity with simple operation and good accuracy. It is proved here that in order to deduce $\Gamma_I(\tau)$ from clipped measurements, one must study, in some cases, the statistical nature of the light. However, it has been very recently shown (Jakeman *et al* 1972) that the so called 'scaling–clipping' method provides, with good approximation, the exact intensity correlation function, regardless of light statistics.

Generalization of the above calculations will be considered in a future work.

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